

T-duality and closed string non-commutative (doubled) geometry

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Abstract

We provide some evidence that closed string coordinates will become non-commutative turning on geometrical fluxes and/or H -field flux background in closed string compactifications. This is in analogy to open string non-commutativity on the world volume of D-branes with B - and F -field background. The class of 3-dimensional backgrounds we are studying are twisted tori (fibrations of a 2-torus over a circle) and their T-dual H -field, 3-form flux backgrounds (T-folds). The spatial non-commutativity arises due to the non-trivial monodromies of the toroidal Kähler resp. complex structure moduli fields, when going around the closed string along the circle direction. In addition we study closed string non-commutativity in the context of doubled geometry, where we argue that in general a non-commutative closed string background is T-dual to a commutative closed string background and vice versa. We also discuss the corresponding spatial uncertainty relations. Finally, in analogy to open string boundary conditions, we also argue that closed string momentum and winding modes define in some sense D-branes in closed string doubled geometry.

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1 Introduction

T-duality is one of the most interesting symmetries in string theory. Most importantly, T-duality is a stringy symmetry in the sense that it arises due to the extended nature of the string in contrast to a point particle. In its simplest and also most transparent manifestation, T-duality arises when one compactifies the closed string on a D-dimensional torus (for a review see [1]). In case of compactification on a one-dimensional circle S^1 of radius R , T-duality is just the invariance of the string theory under the inversion of the radius

$$T: \quad R \longleftrightarrow \frac{\alpha'}{R}. \quad (1)$$

In fact, this symmetry acts on the string spectrum exchanging momentum modes with momenta $p = M/R$ by closed string winding modes of dual momenta $\tilde{p} = (\alpha')^{-1}NR$, i.e. T-duality acts on the integer momentum and winding numbers simply as

$$T: \quad M \longleftrightarrow N. \quad (2)$$

Since this symmetry has a self-dual radius,

$$R_c = \sqrt{\alpha'} = l_s, \quad (3)$$

it follows that the stringy moduli space of inequivalent circle compactifications is given by the interval

$$R \geq R_c. \quad (4)$$

In other words, a compact space of size $R > R_c$ is completely indistinguishable from a compact space with size $R < R_c$, and hence R_c is the shortest possible radius that characterizes the closed string compactification on a circle.

The emergence of a minimal distance in string compactification means that the notion of classical geometry breaks down at distances around $R \sim R_c$. This concept of a minimal observable distance in string theory arises also in a different context, namely in the context of high energy string scattering [2]. Here, it was argued that string high energy scattering experiments leads to a modification of Heisenberg's uncertainty relation of the following form:

$$\Delta x \simeq \frac{1}{\Delta p} + \alpha' \Delta p. \quad (5)$$

It again follows that with a string scattering experiment one cannot resolve distances smaller $\Delta x_{\min} = \sqrt{\alpha'}$, i.e.

$$\Delta x \geq \sqrt{\alpha'}. \quad (6)$$

So far, closed string non-commutativity with respect to the space coordinates was not much discussed in the literature (see e.g. [3, 4]). Non-commutative string

geometry was mainly established for open strings [5, 6, 7, 8, 9]. Specifically it was shown that for open strings with mixed Dirichlet-Neumann boundary conditions on Dp-branes with non-vanishing B -field or gauge field F -background, the open string coordinates at the two ends of the open string, i.e. at $\sigma = 0, \pi$, are non-commutative:

$$[X_1(\tau, \sigma), X_2(\tau, \sigma')]_{\sigma, \sigma'=0, \pi} = i\Theta. \quad (7)$$

The deformation parameter Θ is determined by the open string background parameter $\mathcal{F}_{ij} = B_{ij} + F_{ij} = \epsilon_{ij}\mathcal{F}$ in the following way:

$$\Theta = \frac{2\pi\alpha'\mathcal{F}}{1 + \mathcal{F}^2} \quad (8)$$

Performing a T-duality transformation, the mixed D-N boundary conditions become purely Neumann, and the Dp-branes with \mathcal{F} -flux become D(p-1)-branes that intersect at a certain angle θ , where $\cot \theta = \mathcal{F}$ [10]. Now after T-duality the dual open string coordinates are fully commutative.²

However, also for an extended object like a closed string, T-duality tells us that there is a minimal length that can be dissolved by the string, and hence space should become fuzzy at the string scale. Therefore for extended closed strings with non-trivial winding numbers one could expect a space non-commutativity of the form

$$[X, Y] \neq 0, \quad (9)$$

corresponding to a spatial uncertainty relation

$$\Delta X \Delta Y \neq 0. \quad (10)$$

On the other hand, a pure momentum state of the form $\exp(pX)$ is still a point like object. Hence space coordinates should be still commutative for a pure momentum spaces. Using T-duality, a momentum state becomes a winding state in the dual space with dual coordinates \tilde{X} , whereas states, which are originally winding modes, become momentum states in the dual geometry. On the basis of this behavior, we could deduce that momentum states possibly see a non-commutative dual space with

$$[\tilde{X}, \tilde{Y}] \neq 0, \quad (11)$$

but for winding states the dual geometry is commutative. So again, like for the open string, T-duality exchanges commutative with non-commutative closed string coordinates. Hence if by some reason the \tilde{X} -space is deformed to be non-commutative, it almost trivially follows that in the T-dual geometry the X -space becomes non-commutative, and vice versa.

²If one wants to construct a full fledged open string compactification on a non-commutative torus resp. on the dual torus with intersecting D-branes, further conditions like the vanishing of 1-loop open string tadpoles have to be satisfied [11].

In this paper we show that the natural framework to discuss closed string non-commutative geometry is indeed T-duality in close relation with doubled geometry [12, 13, 14] resp. with doubled field theory [15, 16, 17, 18]; here the usual closed string position space, denoted by string coordinates X^i , is enlarged to contain in addition also the dual closed string coordinates \tilde{X}^i . We will give some evidence that for certain geometrical backgrounds, namely for geometric flux compactifications on twisted tori [19, 20, 21, 22, 23, 24, 25, 26], space with coordinates becomes non-commutative. In fact, these backgrounds are in general T-dual to non-geometric compactifications with a non-trivial H -field background. The main result of this paper will be to give evidence that a closed string on a curved space (3-dimensional torus) with a non-trivial geometrical flux and/or H -flux sees a non-commutative geometry along two directions, which is determined by the (dual) momentum p^3 in the remaining third direction:

$$[X^1, X^2] \sim 2i(l_s)^3 p^3. \quad (12)$$

Therefore for non-vanishing (dual) momenta in the third direction, there will be a spatial uncertainty in the other two directions.

In the appendix we will briefly review the closed string boundary conditions in doubled (X, \tilde{X}) -geometry and the action of T-duality. Momentum resp. winding closed strings are analogous to open strings with Neumann resp. Dirichlet bound conditions; hence we will argue that the choice of closed string boundary conditions also defines a kind of brane in the doubled geometry.

2 Closed string doubled geometry, T-duality and non-commutativity

In this section we recall some known and basic facts about closed strings moving on a compact torus, where we will add a few more remarks about the analogy between open and closed strings in the appendix. First consider the most simple case of a closed string moving on a circle with radius R . Its most general mode expansion has the form

$$X(\tau, \sigma) = X_L(\tau + \sigma) + X_R(\tau - \sigma), \quad (13)$$

$$\begin{aligned} X_L(\tau + \sigma) &= \frac{x}{2} + p_L(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in(\tau + \sigma)}, \\ X_R(\tau - \sigma) &= \frac{x}{2} + p_R(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n e^{-in(\tau - \sigma)}, \end{aligned} \quad (14)$$

where the left- and right-moving momenta are defined as

$$\begin{aligned} p_L &= \frac{1}{2} \left(\frac{M}{R} + (\alpha')^{-1} N R \right), \\ p_R &= \frac{1}{2} \left(\frac{M}{R} - (\alpha')^{-1} N R \right). \end{aligned} \quad (15)$$

Here, the integers M are the Kaluza-Klein momenta, whereas the integers N denote the winding numbers of the closed string around the circle S^1 . The quantized momentum variables p and the dual momenta \tilde{p} are then given as

$$p = p_L + p_R = \frac{M}{R}, \quad \tilde{p} = p_L - p_R = (\alpha')^{-1} N R. \quad (16)$$

T-duality (see eqs.(1) and (2)) acts in the well-known fashion on the momenta:

$$T: \quad p \longleftrightarrow \tilde{p}, \quad p_L \longleftrightarrow p_L, \quad p_R \longleftrightarrow -p_R. \quad (17)$$

Just as for the canonical momenta, we can also introduce a dual space coordinate $\tilde{X}(\tau, \sigma) = X_L - X_R$. The pair (X, \tilde{X}) are the coordinates of so-called doubled geometry. So finally, T-duality acts on the position variables as

$$T: \quad X \longleftrightarrow \tilde{X}, \quad X_L \longleftrightarrow X_L, \quad X_R \longleftrightarrow -X_R. \quad (18)$$

All this can easily be generalized to the compactification of the closed string on a d -dimensional torus T^d . Here we considering a constant metric background G_{ij} as well as a constant background for the antisymmetric tensor field B_{ij} . The canonical left- and right-moving momenta contain also the B-field background and have the following form:

$$p_{L,R}^i = \frac{\alpha'}{2} \left(G_{ij} M^j \pm \frac{1}{\alpha'} (G^{ij} \mp B^{ij}) N_j \right) \quad (19)$$

Then the T-duality symmetries are given by $SO(d, d; \mathbb{Z})$ transformations.

For non-constant backgrounds, one can also apply T-duality in the x -direction, assuming that the background does not depend on the x -coordinate. Then using the Buscher rules [27, 28], T-duality in the x direction provides the following new background:

$$\tilde{G}_{xx} = \frac{1}{G_{xx}}, \quad \tilde{G}_{xi} = -\frac{B_{xi}}{G_{xx}}, \quad \tilde{B}_{xi} = -\frac{G_{xi}}{G_{xx}} \quad (20)$$

$$\tilde{G}_{ij} = G_{ij} - \frac{G_{xi} G_{xj} - B_{xi} B_{xj}}{G_{xx}} \quad (21)$$

$$\tilde{B}_{ij} = B_{ij} - \frac{G_{xi} B_{xj} - B_{xi} G_{xj}}{G_{xx}} \quad (22)$$

$$e^{\tilde{\phi}} = \frac{e^{\phi}}{\sqrt{G_{xx}}} \quad (23)$$

This basically means that a flux background with non-vanishing B_{xi} is T-dualized into a purely geometric back ground with off-diagonal metric G_{xi} and vice versa. Switching from B to H , a non-vanishing H_{xyz} gets T-dualized along the x direction in a metric background, which we call $G_{yz} = f_{yz}^x$

$$H_{xyz} \xrightarrow{T_x} f_{yz}^x. \quad (24)$$

Since these metric components arise from the H -flux after T-duality, one often calls the f_{yz}^x 's metric or also geometric fluxes. As we will discuss in the next chapter, they often appear in twisted tori compactifications where they correspond to a certain, underlying algebraic structure.

In the following we are mostly interested in the case $d = 2$, the compactification on a two-dimensional torus T^2 , which is defined by the two vectors $e_1 = R_1$ and $e_2 = R_2 e^{i\alpha}$, and with additional B -field background $B_{ij} = \epsilon_{ij} B$. This background is then conveniently characterized by two complex parameters: first the complex structure of the torus, denoted by τ , and second by the complexified Kähler parameter, denoted by ρ :

$$\begin{aligned} \tau &= \frac{e_2}{e_1} = \frac{R_2}{R_1} e^{i\alpha}, \\ \rho &= B + i R_1 R_2 \sin \alpha. \end{aligned} \quad (25)$$

The full T-duality group is given by all possible $SO(2, 2; \mathbb{Z})$ transformations. First, it contains the global diffeomorphisms of T^2 given by $SL(2, \mathbb{Z})_\tau$ modular transformations

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}. \quad (26)$$

Second the discrete shifts in B , $B \rightarrow B + n$, together with the overall T-duality transformation, $\rho \rightarrow -1/\rho$, in the x_1 - and in x_2 -directions generate the target space modular group $SL(2, \mathbb{Z})_\rho$ acting as

$$\rho \rightarrow \frac{a\rho + b}{c\rho + d}. \quad (27)$$

Finally, the mirror symmetry, i.e. the T-duality in x_1 -direction is nothing else then the exchange

$$\tau \leftrightarrow \rho. \quad (28)$$

Clearly this transformation exchanges the B -field with $\Re\tau$, which is just proportional to the off-diagonal metric component.

It is now also straightforward to work out the action of the various target space duality transformations on the two-dimensional string coordinates of T^2 . Introducing complex coordinates

$$X = X^1 + iX^2 = X_L + X_R, \quad (29)$$

with $X_L = X_L^1 + iX_L^2$ and likewise for X_R , we see that the geometric $SL(2, \mathbb{Z})_\tau$ transformations act as left-right symmetric transformations on the complex coordinates. E.g. one obtains that

$$\begin{aligned} \tau \rightarrow -1/\tau : \quad & X_L \rightarrow e^{i\theta} X_L, \quad \theta = -\pi/2, \\ & X_R \rightarrow e^{i\theta} X_R. \end{aligned} \quad (30)$$

This is nothing else than a \mathbb{Z}_4 transformation on the coordinates:

$$X_1 \rightarrow X_2, \quad X_2 \rightarrow -X_1. \quad (31)$$

Other elements of $SL(2, \mathbb{Z})_\tau$ like the order three transformation $\tau \rightarrow -1/(\tau + 1)$ act as symmetric \mathbb{Z}_6 rotations on the left- and right-moving coordinates.

On the other hand, the target space duality transformations in $SL(2, \mathbb{Z})_\rho$ basically act as asymmetric rotations on X_L and X_R [29]. E.g. consider the overall duality transformation $\rho \rightarrow -1/\rho$. It acts as follows:

$$\begin{aligned} \rho \rightarrow -1/\rho : \quad & X_L \rightarrow e^{i\theta} X_L, \quad \theta = -\pi/2, \\ & X_R \rightarrow e^{-i\theta} X_R. \end{aligned} \quad (32)$$

Finally let us also consider the dual geometry obtained by the mirror transformation $\tau \leftrightarrow \rho$, i.e. T-duality in the x_1 direction. Here one obtains the following asymmetric rotation:

$$\begin{aligned} \tau \leftrightarrow \rho : \quad & X_L \leftrightarrow \tilde{X}_L = X_L, \\ & X_R \leftrightarrow \tilde{X}_R = -\tilde{X}_R. \end{aligned} \quad (33)$$

The action of the $SL(2, \mathbb{Z})_\tau$ transformations now become asymmetric rotations on the dual coordinates,

$$\begin{aligned} \tau \rightarrow -1/\tau : \quad & \tilde{X}_L \rightarrow e^{i\theta} \tilde{X}_L, \quad \theta = -\pi/2, \\ & \tilde{X}_R \rightarrow e^{-i\theta} \tilde{X}_R. \end{aligned} \quad (34)$$

where the $SL(2, \mathbb{Z})_\rho$ transformations act symmetrically:

$$\begin{aligned} \rho \rightarrow -1/\rho : \quad & \tilde{X}_L \rightarrow e^{i\theta} \tilde{X}_L, \quad \theta = -\pi/2, \\ & \tilde{X}_R \rightarrow e^{i\theta} \tilde{X}_R. \end{aligned} \quad (35)$$

Although already basically well known, all these transformation properties of the left- and right-moving coordinates will turn out to be important for the closed string non-commutativity. But before we analyze in more detail how non-commutativity can arise, let us make the following general observations about

closed string non-commutativity. Suppose one finds in some background the following commutation relations between left- and right-moving complex string coordinates:

$$[X_L, \bar{X}_L] = -[X_R, \bar{X}_R] = \Theta, \quad [X_L, X_R] = [X_L, \bar{X}_R] = 0. \quad (36)$$

As we will show in section (3.2) for the case of the shifts in σ , these commutators correspond to the symmetric rotations of the form eq.(30), which are induced by the $SL(2, \mathbb{Z})_\tau$ modular transformations. They imply the following commutation relation between the left- and right-moving real coordinates:

$$[X_L^1, X_L^2] = -[X_R^1, X_R^2] = i\Theta/2. \quad (37)$$

Finally going to the position space coordinates $X^{1,2} = X_L^{1,2} + X_R^{1,2}$, one then obtains

$$[X^1, X^2] = [X_L^1 + X_R^1, X_L^2 + X_R^2] = 0. \quad (38)$$

So in position space this background corresponds to a commutative geometry.

Now we switch to the dual coordinates with respect to the x_1 direction, which is equivalent to perform a T-duality transformation in the x_1 direction. As it is well known, for type II backgrounds, this T-duality transformation exchanges the type IIA string with the type IIB string and vice versa. As discussed this acts as $X_R^1 \rightarrow -X_R^1$. Hence the commutator eq.(38) is replaced by the following expression

$$[\tilde{X}^1, X^2] = [X_L^1 - X_R^1, X_L^2 + X_R^2] = i\Theta. \quad (39)$$

So we see that the dual space coordinates, or equivalently the T-dual background, are now non-commutative. This justifies our claim in the introduction. In the next chapter we will demonstrate this behavior for an explicit example.

In fact, for the case of the shifts in σ , the non-commutative coordinates of the T-dual space are precisely associated to the asymmetric rotations eq.(32) induced by the duality transformation $SL(2, \mathbb{Z})_\rho$. They will induce to the following commutation relations between the left- and right-moving complex coordinates:

$$[X_L, \bar{X}_L] = [X_R, \bar{X}_R] = \Theta, \quad [X_L, X_R] = [X_L, \bar{X}_R] = 0. \quad (40)$$

It now follows that

$$[X^1, X^2] = [X_L^1 + X_R^1, X_L^2 + X_R^2] = i\Theta. \quad (41)$$

In case of constant metric and constant B-field one can easily show that all coordinates and also all dual coordinates are fully commuting. Therefore we have to pose the question, how is it possible to make the X -space (or the \tilde{X} -space) non-commutative by turning on certain background fields, in analogy to the non-commutative open string space when turning on a B-field background on the open string D-branes? For that we need also closed string states with kind of mixed

boundary conditions. In other words, we will need twisted closed string sectors, in analogy to those open strings, which are stretching between two D-branes with different \mathcal{F} -field backgrounds. The necessary twisting will be achieved by compactifying the closed string on a twisted torus with geometric fluxes turned on. These geometric fluxes will still lead to commutative X -space, in case of shifts in σ (see section 3.2). However for shifts in τ the geometric fluxes will lead to a non-commutative coordinate space (see section 3.3). After T-duality the geometric fluxes will be dualized into a non-constant B -field background, i.e. into a non-trivial H -field. Then, as we will discuss, the dual geometry with non-trivial H -field will be non-commutative for the σ shifts. As emphasized already, all this is most naturally discussed in the context of T-folds resp. of doubled geometry. On this space the T-duality group $SO(d, d)$ has a very natural action. More generally, under T-duality the H -field background turns into various geometries like Taub-Nut spaces or Eguchi Hanson instantons.

3 The twisted torus and the T-dual H -field geometry

3.1 Monodromies and duality transformations

Let us demonstrate the closed string non-commutativity and the associated T-duality transformation rules by an explicit example, namely that of a twisted T^3 with geometrical fluxes, which is dual to a T^3 with H -flux. These backgrounds are closely related to Scherk-Schwarz compactifications and freely acting orbifold compactifications, discussed e.g. in [30, 31, 32, 33, 34, 35, 36]. Note that this background does not necessary satisfy the supersymmetry conditions. This is not a problem, however, as we only use this as an illustrative example and one could e.g. fiber this T^3 over something else to get a good string background.

To start, take (x^1, x^2, x^3) as the coordinates on the T^3 . First, we consider the twisted torus without B -field background. (For superstrings, this would be the relevant background in type IIA). Its metric has the form:

$$ds^2 = \frac{1}{\Im\tau} |dx^1 + \tau(z)dx^2|^2 + (dx^3)^2, \quad B = H = 0. \quad (42)$$

In general, the complex structure τ of the T^2 is not a constant, but it is rather a non-trivial function of circle coordinate x^3 : $\tau(x^3) = f(x^3)$. One can easily picture this space as a T^2 in the (x^1, x^2) directions fibered over an S^1 in the x^3 direction. As one goes around the S^1 base (we assume that the S^1 has unit radius),

$$x^3 \rightarrow x^3 + 2\pi, \quad (43)$$

the fiber T^2 undergoes a transformation

$$\tau(x^3) \rightarrow \tau(x^3 + 2\pi). \quad (44)$$

If we want to end up with an equivalent fiber after going around the S^1 , we need to ensure that this is an $SL(2, \mathbb{Z})_\tau$ transformation, i.e.

$$\tau(x^3 + 2\pi) = \frac{a\tau(x^3) + b}{c\tau(x^3) + d}. \quad (45)$$

These transformation define the monodromy properties of the torus fibration.

Now we perform a T-duality transformation on the x^1 direction: this yields the background of a rectangular T^3 with non-trivial B - and H -field. (In type II compactifications, one now goes from IIA to IIB at the same time.) Specifically, this background is simply obtained by replacing the complex structure $\tau(x^3)$ of the twisted torus by the Kähler parameter $\rho(x^3)$ of the dual geometry, i.e. $\rho(x^3) = f(x^3)$. This background is still a non-trivial fibration, since the volume of the T^2 as well as the B -field³ now vary over the S^1 :

$$\text{Vol}^{T^2}(x^3) = \Im \rho(x^3), \quad B(x^3) = \Re \rho(x^3), \quad H(x^3) = \frac{d}{dx^3} B(x^3). \quad (46)$$

Again we have to require that the fibration is such that going around the S^1 circle one changes the Kähler parameter ρ only by a target space T-duality transformation, i.e..

$$\rho(x^3 + 2\pi) = \frac{a\rho(x^3) + b}{c\rho(x^3) + d}. \quad (47)$$

In general, these backgrounds are not anymore geometric manifolds, since global diffeomorphisms only close up to T-duality transformations. Therefore they are called T-folds [12].

Let us now discuss a few examples of monodromies that imply certain choices for the function $f(x^3)$. This discussion is of course equally valid for the twisted torus and the T-dual H -field background.

(i) *Trivial monodromy:*

Here $f(x^3 + 2\pi) = f(x^3)$. The simplest choice obviously is $f(x^3) = \text{const.}$, but also other choices of period functions are possible. For $f(x^3) = \text{const.}$, the H -field of course vanishes.

(ii) *Parabolic monodromies:*

Parabolic monodromies are basically generated by discrete shift in the function $f(x^3)$:

$$f(x^3) = \frac{1}{2\pi} H x^3 + \text{const.}, \quad H \in \mathbb{Z} \quad (48)$$

These monodromies are of infinite order. On the (IIA) twisted torus (here the so-called nilmanifold), the shift $x^3 \rightarrow x^3 + 2\pi$ has to be followed by a shift in the

³With this choice of B -field we have picked a gauge where $B_{x^1 x^2}(x^3) \neq 0$. In addition, as we will see later, the H -field is quantized such that the following condition is satisfied: $\int_{T^3} H = N$.

coordinate x^1 :

$$x^1 \rightarrow x^1 + Hx^2, \quad x^2 \rightarrow x^2. \quad (49)$$

In the dual (IIB) geometry on the flat 3-torus, going around the circle S^1 means that we shift the linear B -field,

$$B = \frac{1}{2\pi} H x^3, \quad (50)$$

by an integer: $B \rightarrow B + H$. The H -field is constant and is given by an integer number.

(iii) *Elliptic monodromies*:

Elliptic monodromies are of finite order. They act as \mathbb{Z}_N -transformations on the T^2 coordinates. Example are the order two transformation $f(x^3 + 2\pi) = -1/f(x^3)$ and the order three transformation $f(x^3 + 2\pi) = -1/(f(x^3) + 1)$.

Let us analyze the first transformation in more detail. For the (IIA) twisted torus, $\tau(x^3) \rightarrow -1/\tau(x^3)$ acts on the torus coordinates x^1 and x^2 as a \mathbb{Z}_4 rotation:

$$x^1 \rightarrow x^2, \quad x^2 \rightarrow -x^1, \quad (51)$$

or written in complex coordinates $x = x^1 + ix^2$:

$$x \rightarrow e^{-2\pi i H} x, \quad H \in \frac{1}{4} + \mathbb{Z}. \quad (52)$$

A function $\tau(x^3)$ with the required monodromy properties is e.g. given as [23]:

$$\tau(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)}. \quad (53)$$

After T-duality (to IIB), we exchange $\tau(x^3)$ with $\rho(x^3)$, and the B -field is given as

$$B(x^3) = \Re \rho(x^3) = \frac{\sin(2Hx^3) - 2\cos(2Hx^3)}{2\sin(2Hx^3) + \cos(2Hx^3) - 3}, \quad (54)$$

and the H -field has the form

$$H(x^3) = H \frac{10 - 12\sin(2Hx^3) - 6\cos(2Hx^3)}{(2\sin(2Hx^3) + \cos(2Hx^3) - 3)^2}. \quad (55)$$

This expression can be expanded for small x^3 (or for small H), where the H -field becomes constant: $H(x^3) = H$. Note that $H(x^3)$ is a periodic function in x^3 with period 4π ,

$$H(x^3 + 4\pi) = H(x^3), \quad (56)$$

i.e. going twice around the circle does not change the H -field. Going once the circle one gets

$$H(x^3 + 2\pi) = -H(x^3), \quad (57)$$

These transformation properties of the H -field just correspond to the \mathbb{Z}_4 order of the considered monodromy transformation.

In addition, also the volume of T^2 is varying over circle in the x^3 -direction:

$$\text{Vol}^{T^2}(x^3) = \Im \rho(x^3) = \frac{1}{1 + \sin^2(Hx^3) - \sin(2Hx^3)}. \quad (58)$$

In fact, going once around the circle exchanges a large T^2 with a small T^2 , which are however equivalent from the closed string point of view because of T-duality in the x^1 - and x^2 -directions. Therefore this background is not a geometrical background in the usual sense, but a T-fold. A similar analysis can be also done for the elliptic monodromy of order three.

3.2 The σ -monodromies and winding non-commutativity

So far we have discussed the monodromy properties of the twisted torus geometry and its dual B -field background geometry. But now we have to remember that we are considering a closed string moving in this background. Therefore we have to consider the closed string boundary conditions in this background. Let us start with the twisted torus geometry. In the circle x^3 -direction, the closed string obeys the standard periodicity conditions

$$X^3(\tau, \sigma + 2\pi) = X^3(\tau, \sigma) + 2\pi N_3, \quad (59)$$

where N_3 is the winding number of the closed string in the x^3 -direction. According to our previous discussion, due to the non-trivial fibration structure the non-trivial boundary conditions in the x^3 direction will also act on the coordinates of the 2-torus, namely on $\tau(x^3)$ resp. on $\rho(x^3)$, in a non-trivial way. As a consequence, depending on which monodromy properties we are considering, one gets shifted or twisted closed string boundary conditions in the $X^{1,2}$ -directions. Specifically, neglecting the momentum and winding zero modes in these two directions, we obtain the following closed string boundary conditions for the three monodromies considered before:

(i) *Trivial monodromy:*

$$X^{1,2}(\tau, \sigma + 2\pi) = X^{1,2}(\tau, \sigma). \quad (60)$$

(ii) *Parabolic monodromies:*

Now the closed string boundary conditions include the following shift in the coordinate x^1 :

$$X^1(\tau, \sigma + 2\pi) = X^1(\tau, \sigma) + N_3 H X^2(\tau, \sigma), \quad X^2(\tau, \sigma) = X^2(\tau, \sigma). \quad (61)$$

(iii) *Elliptic monodromies:*

Here, the transformation $\sigma \rightarrow \sigma + 2\pi$ acts as a rotation in the $X^{1,2}$ -directions. Namely one obtains $(X(\tau, \sigma) = X^1(\tau, \sigma) + iX^2(\tau, \sigma))$:

$$X(\tau, \sigma + 2\pi) = e^{-2\pi i N_3 H} X(\tau, \sigma). \quad (62)$$

Note that in both cases the shift parameter or resp. the rotation angle is given by the product $N_3 H$, i.e. non-trivial boundary conditions require a non-zero winding number N_3 in connection with a non-vanishing H -field.

As a specific example, we now want to analyze in more detail the elliptic monodromy corresponding to the modular transformation $\tau(x^3) \rightarrow -1/\tau(x^3)$; it leads to the \mathbf{Z}_4 twisted boundary conditions displayed in eq.(62) with $N_3 H \in 1/4 + \mathbb{Z}$. Switching to complex, left- and right-moving closed string coordinates X_L and X_R , we are dealing with the following left-right symmetric closed string boundary conditions for the twisted torus geometry:

$$\begin{aligned} X_L(\tau, \sigma + 2\pi) &= e^{i\theta} X_L(\tau, \sigma), \quad \theta = -2\pi N_3 H, \\ X_R(\tau, \sigma + 2\pi) &= e^{i\theta} X_R(\tau, \sigma). \end{aligned} \quad (63)$$

The complex string coordinates, which precisely possess these boundary conditions like a symmetric orbifold [37, 38, 39, 40], have the following twisted mode expansion (this mode expansion is quite similar to the mode expansion of open strings on intersecting D1-branes – see the appendix):

$$\begin{aligned} X_L(\tau + \sigma) &= i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{n - \nu} \alpha_{n-\nu} e^{-i(n-\nu)(\tau+\sigma)}, \quad \nu = \frac{\theta}{2\pi} = -N_3 H, \\ X_R(\tau - \sigma) &= i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{n + \nu} \tilde{\alpha}_{n+\nu} e^{-i(n+\nu)(\tau-\sigma)}, \end{aligned} \quad (64)$$

Then $\bar{X}_L(\tau, \sigma)$ is given as

$$\bar{X}_L(\tau + \sigma) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{n - \nu} \bar{\alpha}_{n-\nu} e^{i(n-\nu)(\tau+\sigma)}, \quad (65)$$

and similarly for $\bar{X}_R(\tau, \sigma)$. Here, the $\alpha_{n-\nu}$ and the $\bar{\alpha}_{n-\nu}$ are complex, twisted oscillators with the following commutation relations

$$[\alpha_{n-\nu}, \bar{\alpha}_{m-\nu}] = (n - \nu) \delta_{n,m}, \quad (66)$$

and likewise for $\tilde{\alpha}_{n+\nu}$. It follows that $X_L^1(\tau, \sigma)$ has the following form,

$$\begin{aligned} X_L^1(\tau + \sigma) &= \frac{X_L(\tau + \sigma) + \bar{X}_L(\tau + \sigma)}{2} \\ &= \frac{i}{2} \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{n - \nu} \left[\alpha_{n-\nu}^1 e^{-i(n-\nu)(\tau+\sigma)} - \alpha_{-(n-\nu)}^1 e^{i(n-\nu)(\tau+\sigma)} \right. \\ &\quad \left. + i \left(\alpha_{n-\nu}^2 e^{-i(n-\nu)(\tau+\sigma)} + \alpha_{-(n-\nu)}^2 e^{i(n-\nu)(\tau+\sigma)} \right) \right], \end{aligned} \quad (67)$$

with $\alpha_{n-\nu} = \alpha_{n-\nu}^1 + i\alpha_{n-\nu}^2$, and where we have used that

$$\alpha_{-(n-\nu)}^* = \bar{\alpha}_{n-\nu}. \quad (68)$$

For $\nu = 0$, eq.(67) collapses to

$$X_L^1(\tau + \sigma) = i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^1 e^{-in(\tau+\sigma)}. \quad (69)$$

Here we have removed the divergent zero-mode term with $n = 0$. Similar expressions hold for $X_L^2(\tau, \sigma)$, $X_R^1(\tau, \sigma)$ and $X_R^2(\tau, \sigma)$.

Using these mode expansions and the commutation relations among the oscillators we can now compute the commutator between two left- or right-moving complex closed string coordinates at equal times τ . One obtains the following result:

$$\begin{aligned} [X_L(\tau, \sigma), \bar{X}_L(\tau, \sigma')] &= \Theta(\sigma, \sigma') = \alpha' \sum_{n \in \mathbb{Z}} \frac{1}{n - \nu} e^{-i(n-\nu)(\sigma-\sigma')}, \\ [X_R(\tau, \sigma), \bar{X}_R(\tau, \sigma')] &= -\Theta(\sigma, \sigma'). \end{aligned} \quad (70)$$

This series can be written in terms of hypergeometric functions as follows ($z = e^{i(\sigma'-\sigma)}$):

$$\Theta(\sigma, \sigma') = \frac{1}{\nu} z^\nu \left({}_2F_1(1, -\nu; 1 - \nu; z) - {}_2F_1(1, \nu; 1 + \nu; z^{-1}) - 1 \right). \quad (71)$$

For $\sigma = \sigma'$, this sum can be explicitly evaluated:⁴

$$\Theta = \alpha' \sum_{n \in \mathbb{Z}} \frac{1}{n - \nu} = \alpha' \pi \cot \theta / 2 = -\alpha' \pi \cot(\pi N_3 H). \quad (72)$$

Using $H = 1/4$, one obtains that $\Theta = -\alpha' \pi$ for winding number $N_3 = 1$ ($\theta = -\pi/2$), whereas $\Theta = 0$ for $N_3 = 2$ ($\theta = -\pi$). Also note that Θ formally diverges for $\theta = 0 \bmod 2\pi$. This infinity arises, since the sum eq.(72) contains the divergent term $1/n$. However for $\theta = 0 \bmod 2\pi$ the $1/n$ -term has to be excluded from the untwisted mode expansion. Taking this into account and summing over all terms with $n \neq 0$, one indeed obtains $\Theta = 0$ for $\theta = 0 \bmod 2\pi$; hence Θ exhibits a discontinuity at $\theta = 0 \bmod 2\pi$.

We see that the non-vanishing commutators between the left-moving coordinates and the right-moving coordinates have opposite signs. Therefore we encounter precisely the situation of eq.(36). It follows that the geometric, twisted torus geometry is commutative:

$$[X^1(\tau, \sigma), X^2(\tau, \sigma)] = [X_L^1 + X_R^1, X_L^2 + X_R^2] = 0. \quad (73)$$

⁴Here one can use that $\psi(x) = \partial_x \ln \Gamma(x)$, $\pi \cot(\pi z) = \psi(1-z) - \psi(z)$, $\psi(1-z) = -\gamma_E - \frac{1}{1-z} - \sum_{n=1}^{\infty} \left(\frac{1}{1-z+n} - \frac{1}{n} \right)$.

However switching to the dual coordinates after a T-duality in the x^1 -direction one obtains a non-trivial commutator:

$$[\tilde{X}^1(\tau, \sigma), X^2(\tau, \sigma)] = [X_L^1 - X_R^1, X_L^2 + X_R^2] = i\Theta. \quad (74)$$

It means that the dual H -field background with varying Kähler parameter becomes non-commutative. This can be explicitly checked by looking at the closed string boundary conditions of the H -field background. The closed string boundary condition $X^3(\tau, \sigma + 2\pi) = X^3(\tau, \sigma) + 2\pi N_3$ in the circle direction now corresponds to the monodromy transformation $\rho \rightarrow -1/\rho$ on the 2-torus. This transformation implies the following left-right asymmetric boundary conditions, in analogy to the boundary conditions of an asymmetric orbifold [41]:

$$\begin{aligned} X_L(\tau, \sigma + 2\pi) &= e^{i\theta} X_L(\tau, \sigma), \quad \theta = -2\pi N_3 H, \\ X_R(\tau, \sigma + 2\pi) &= e^{-i\theta} X_R(\tau, \sigma). \end{aligned} \quad (75)$$

They are consistent with the following mode expansion (this mode expansion is quite similar to the mode expansion of open strings on D2-branes with \mathcal{F} -flux – see the appendix):

$$\begin{aligned} X_L(\tau + \sigma) &= i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{n - \nu} \alpha_{n-\nu} e^{-i(n-\nu)(\tau+\sigma)}, \quad \nu = \frac{\theta}{2\pi} \\ X_R(\tau - \sigma) &= i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{n - \nu} \tilde{\alpha}_{n-\nu} e^{-i(n-\nu)(\tau-\sigma)}, \end{aligned} \quad (76)$$

Then one obtains in the dual space that

$$[X_L(\tau, \sigma), \bar{X}_L(\tau, \sigma')] = [X_R(\tau, \sigma), \bar{X}_R(\tau, \sigma')] = \Theta(\sigma, \sigma'), \quad (77)$$

and finally

$$[X^1(\tau, \sigma), X^2(\tau, \sigma)] = i\Theta. \quad (78)$$

So, the T-fold with H -flux becomes non-commutative, the non-commutativity parameter Θ originates from the winding number N_3 in the third direction.

3.3 The τ -monodromies and momentum non-commutativity

It is now straightforward to consider those monodromies on the (dual) circle on x^3 -direction, which correspond to the (not so conventional) boundary conditions with respect to the world sheet coordinate τ (see also the appendix). Specifically, consider, instead of an winding string in the third direction, a string with momentum number M_3 . Then, going through the same kind of arguments and computations as in the previous subsection, one obtains from the "periodicity" condition with respect to τ ,

$$X^3(\tau + 2\pi, \sigma) = X^3(\tau, \sigma) + 2\pi M_3, \quad (79)$$

the following commutators for the twisted torus with geometric fluxes:

$$\begin{aligned}
[X_L(\tau, \sigma), \bar{X}_L(\tau, \sigma)] &= \tilde{\Theta} = -\alpha' \pi \cot(\pi M_3 H), \\
[X_R(\tau, \sigma), \bar{X}_R(\tau, \sigma)] &= \tilde{\Theta}, \\
[X_L^1(\tau, \sigma), X_L^2(\tau, \sigma)] &= i\tilde{\Theta}/2, \\
[X_R^1(\tau, \sigma), X_R^2(\tau, \sigma)] &= i\tilde{\Theta}/2.
\end{aligned} \tag{80}$$

Hence we now obtain the following non-vanishing commutator between the space coordinates X^1 and X^2 :

$$[X^1(\tau, \sigma), X^2(\tau, \sigma)] = i\tilde{\Theta}. \tag{81}$$

On the other hand, for the dual geometry, namely for the T-fold with H -flux one now gets

$$\begin{aligned}
[X_L(\tau, \sigma), \bar{X}_L(\tau, \sigma)] &= \tilde{\Theta} = -\alpha' \pi \cot(\pi M_3 H), \\
[X_R(\tau, \sigma), \bar{X}_R(\tau, \sigma)] &= -\tilde{\Theta}, \\
[X_L^1(\tau, \sigma), X_L^2(\tau, \sigma)] &= i\tilde{\Theta}/2, \\
[X_R^1(\tau, \sigma), X_R^2(\tau, \sigma)] &= -i\tilde{\Theta}/2, \\
[X^1(\tau, \sigma), X^2(\tau, \sigma)] &= 0.
\end{aligned} \tag{82}$$

So as before, a non-commutative background is T-dual to a commutative background and vice versa.

3.4 Combining winding and momentum non-commutativity

Now consider the general case with non-vanishing momentum and winding numbers in the third direction, i.e. $M_3, N_3 \neq 0$, which leads to two non-trivial monodromies on the (self-dual) circle in x^3 -direction. Then combining the results from the previous two subsections, we obtain for the twisted torus with geometrical fluxes the following commutators:

$$\begin{aligned}
[X_L(\tau, \sigma), \bar{X}_L(\tau, \sigma)] &= -\alpha' \pi \left(\cot(\pi M_3 H) + \cot(\pi N_3 H) \right), \\
[X_R(\tau, \sigma), \bar{X}_R(\tau, \sigma)] &= -\alpha' \pi \left(\cot(\pi M_3 H) - \cot(\pi N_3 H) \right), \\
[X_L^1(\tau, \sigma), X_L^2(\tau, \sigma)] &= -i \frac{\alpha' \pi}{2} \left(\cot(\pi M_3 H) + \cot(\pi N_3 H) \right), \\
[X_R^1(\tau, \sigma), X_R^2(\tau, \sigma)] &= -i \frac{\alpha' \pi}{2} \left(\cot(\pi M_3 H) - \cot(\pi N_3 H) \right), \\
[X^1(\tau, \sigma), X^2(\tau, \sigma)] &= -i \alpha' \pi \cot(\pi M_3 H).
\end{aligned} \tag{83}$$

Likewise we obtain for the T-fold with H -flux:

$$[X_L(\tau, \sigma), \bar{X}_L(\tau, \sigma)] = -\alpha' \pi \left(\cot(\pi M_3 H) + \cot(\pi N_3 H) \right),$$

$$\begin{aligned}
[X_R(\tau, \sigma), \bar{X}_R(\tau, \sigma)] &= -\alpha' \pi \left(-\cot(\pi M_3 H) + \cot(\pi N_3 H) \right), \\
[X_L^1(\tau, \sigma), X_L^2(\tau, \sigma)] &= -i \frac{\alpha' \pi}{2} \left(\cot(\pi M_3 H) + \cot(\pi N_3 H) \right), \\
[X_R^1(\tau, \sigma), X_R^2(\tau, \sigma)] &= -i \frac{\alpha' \pi}{2} \left(-\cot(\pi M_3 H) + \cot(\pi N_3 H) \right), \\
[X^1(\tau, \sigma), X^2(\tau, \sigma)] &= -i \alpha' \pi \cot(\pi N_3 H).
\end{aligned} \tag{84}$$

So we see that now both backgrounds are non-commutative in X -space and also in \tilde{X} -space. Hence the situation is now symmetric with respect to T-duality.

4 Non-commutative geometry and (dual) momentum dependent space uncertainty

So far, the right hand sides of the commutators of X^1 and X^2 were determined by the winding and momentum numbers N_3 and M_3 . However these two numbers are nothing else than the discrete eigenvalues of the canonical momentum operator P^3 and of the dual momentum operator \tilde{p}^3 of the closed string on the circle in the third direction.⁵ Therefore we now conjecture that the full non-commutative algebraic structure is obtained by replacing the momentum number M_3 by the (zero mode) momentum operator p^3 and the winding number N_3 by the dual momentum operator \tilde{p}^3 in all above equations:

$$M_3 \equiv \sqrt{\alpha'} p^3, \quad N_3 \equiv \sqrt{\alpha'} \tilde{p}^3. \tag{85}$$

Furthermore expanding the cot-function (around $\pi/4$) with respect to these operators, we obtain at linear order in the (dual) momenta the following refined commutation relations, where the right hand sides of the commutators are now given in terms of (dual) momentum operators. First for the twisted torus with geometrical fluxes we get ($\alpha' = l_s^2$):

$$\begin{aligned}
[X_L(\tau, \sigma), \bar{X}_L(\tau, \sigma)] &= 2l_s^3 \pi^2 H (p^3 + \tilde{p}^3) = 2l_s^3 \pi^2 H p_L^3, \\
[X_R(\tau, \sigma), \bar{X}_R(\tau, \sigma)] &= 2l_s^3 \pi^2 H (p^3 - \tilde{p}^3) = 2l_s^3 \pi^2 H p_R^3, \\
[X_L^1(\tau, \sigma), X_L^2(\tau, \sigma)] &= il_s^3 \pi^2 H (p^3 + \tilde{p}^3) = il_s^3 \pi^2 H p_L^3, \\
[X_R^1(\tau, \sigma), X_R^2(\tau, \sigma)] &= il_s^3 \pi^2 H (p^3 - \tilde{p}^3) = il_s^3 \pi^2 H p_R^3, \\
[X^1(\tau, \sigma), X^2(\tau, \sigma)] &= 2il_s^3 \pi^2 H p^3.
\end{aligned} \tag{86}$$

We see that the commutator between two space coordinates is now given in terms of the momentum operator p^3 . Here, the constant H on the right hand side just

⁵This section, which is not in the first version of our paper, was partly inspired by the recent work [42], namely to argue that the commutator of two space coordinates is not a c-number but rather an operator.

corresponds to the geometrical flux of the twisted torus. We will further discuss this result and the corresponding uncertainty relations at the end of this section.

Before we come to this discussion, let us determine the analogous commutators for the dual geometry, the T-fold with H -flux. Here we get:

$$\begin{aligned}
[X_L(\tau, \sigma), \bar{X}_L(\tau, \sigma)] &= 2l_s^3 \pi^2 H(p^3 + \tilde{p}^3) = 2l_s^3 \pi^2 H p_L^3, \\
[X_R(\tau, \sigma), \bar{X}_R(\tau, \sigma)] &= -2l_s^3 \pi^2 H(p^3 - \tilde{p}^3) = -2l_s^3 \pi^2 H p_R^3, \\
[X_L^1(\tau, \sigma), X_L^2(\tau, \sigma)] &= il_s^3 \pi^2 H(p^3 + \tilde{p}^3) = il_s^3 \pi^2 H p_L^3, \\
[X_R^1(\tau, \sigma), X_R^2(\tau, \sigma)] &= -il_s^3 \pi^2 H(p^3 - \tilde{p}^3) = -il_s^3 \pi^2 H p_R^3, \\
[X^1(\tau, \sigma), X^2(\tau, \sigma)] &= 2il_s^3 \pi^2 H \tilde{p}^3.
\end{aligned} \tag{87}$$

Now the constant H on the right hand side corresponds to the H -flux of the T-fold. Note that these two sets of equations precisely have the correct transformation properties under T-duality.

Equations (86) and (87) can be neatly combined for backgrounds that possess both geometrical and H -fluxes. Denote the geometrical flux by f^{123} and the H -flux by H^{123} , which are T-dual to each other (see eq.(24)), we can write these equations as

$$\begin{aligned}
[X_L(\tau, \sigma), \bar{X}_L(\tau, \sigma)] &= 2l_s^3 \pi^2 (f^{123} + H^{123}) p_L^3, \\
[X_R(\tau, \sigma), \bar{X}_R(\tau, \sigma)] &= 2l_s^3 \pi^2 (f^{123} - H^{123}) p_R^3, \\
[X_L^1(\tau, \sigma), X_L^2(\tau, \sigma)] &= il_s^3 \pi^2 (f^{123} + H^{123}) p_L^3, \\
[X_R^1(\tau, \sigma), X_R^2(\tau, \sigma)] &= il_s^3 \pi^2 (f^{123} - H^{123}) p_R^3, \\
[X^1(\tau, \sigma), X^2(\tau, \sigma)] &= 2il_s^3 \pi^2 (f^{123} p^3 + H^{123} \tilde{p}^3).
\end{aligned} \tag{88}$$

Now let us discuss the physical meaning of these equations. First note that we have two sets of commutators in doubled geometry, namely one for the left-movers and another one for the right-movers, and that they are fully consistent with T-duality (in our example in x^1 -direction). Second, looking at the last equations in (86) or in (88), we see that in a curved gravitational background with non-trivial geometrical flux, the commutator of the two closed string coordinates X^1 and X^2 is proportional to the product of the geometrical flux times the momentum in the third direction:

$$[X^1, X^2] = 2il_s^3 \pi^2 f^{123} p^3. \tag{89}$$

Therefore from this commutation relation we can derive the following uncertainty relation for the coordinates X^1 and X^2 :

$$(\Delta X^1)^2 (\Delta X^2)^2 \geq \pi^4 l_s^6 (f^{123})^2 \langle p^3 \rangle^2. \tag{90}$$

This means that the uncertainty in X^1 and X^2 is proportional to the expectation value of the momentum operator in the third direction. So, for small momenta, this uncertainty almost vanishes. However for high momenta of the order $\langle p^3 \rangle \sim$

l_s^{-1} in the third direction of the space, the spatial uncertainty in the other two directions is of the order l_s^2 . In other words, the position of a string, that moves with high velocity in the third direction, cannot not be precisely determined in the other two directions. This is what one could perhaps expect in a theory of quantum gravity on a curved space: space becomes non-commutative, i.e. fuzzy, at high energies, but is almost commutative at low energies. Also note, since p^3 and x^3 have standard, canonical Heisenberg commutation relation,

$$[p^3, X^3] = -i, \quad (91)$$

we get

$$[[X^1, X^2], X^3] = 2\pi^2 f^{123} l_s^3. \quad (92)$$

Hence the uncertainty of the product between the commutator $[X^1, X^2]$ and X^3 is just the usual Heisenberg uncertainty relation in the third direction, and is given by

$$\begin{aligned} (\Delta[X^1, X^2])^2 (\Delta X^3)^2 &= \pi^4 (f^{123})^2 l_s^6 (\Delta p^3)^2 (\Delta X^3)^2 \\ &\geq \pi^4 (f^{123})^2 l_s^6. \end{aligned} \quad (93)$$

Of course, one can derive similar uncertainty relations for H -flux backgrounds, where one replaces the momentum p^3 by the dual momentum \tilde{p}^3 .

5 Summary & Outlook

Let us summarize this paper with a few remarks, questions and also with a few speculations:

- The main result of this paper is that we gave some evidence that a closed string on a curved space (3-dimensional torus) with a non-trivial geometrical flux f^{123} and H -field H^{123} sees a non-commutative geometry along two directions, which is determined by the (dual) momentum in the remaining third direction:

$$[X^1(\tau, \sigma), X^2(\tau, \sigma)] = 2il_s^3 \pi^2 (f^{123} p^3 + H^{123} \tilde{p}^3). \quad (94)$$

In a more symmetric situation between all three coordinates, we can combine this non-commutative closed string geometry together with the standard Heisenberg algebra of coordinates and momenta, and we are led to the conjecture that the following interesting algebraic structure (here in three dimensions) is underlying closed string non-commutative geometry:

$$[x^i, x^j] \sim ic \epsilon^{ijk} p^k, \quad [p^i, p^j] \sim ic' \epsilon^{ijk} p^k, \quad [x^i, p^j] = i \delta^{ij}, \quad (95)$$

where c is a constant proportional to l_s^3 , and c' is a constant proportional to l_s^{-1} ,

- We made several assumptions in our computation; first, the mode expansion of the coordinates we used is basically the one of a free, twisted boson. Of course the metric and B -field background depend on the coordinates. So in principle we are dealing with an interacting σ -model. Therefore our ansatz should be regarded as lowest order approximation in the H -field. This seems to be reasonable, because the back reaction of the H -field, namely its contribution to the β -function is quadratic in H . Secondly, in our computation we have assumed a particular gauge, in which the B -field components are only along the x^1 - and x^2 -directions of the 3-dimensional torus; in addition the B -field only depends on the third coordinate x^3 .
- Studying the $SU(2)$ WZW model with a quantized H -field, the authors of [42] (see also [43]) give some evidence that the corresponding closed string geometry is not only non-commutative, but is even non-associative. The basic object, studied in [42], is the 3-bracket

$$[[X^1, X^2], X^3] + \text{cyclic} . \quad (96)$$

We can confirm the non-associativity of [42], namely that for the two dual backgrounds, we studied in the paper, the 3-bracket is indeed non-vanishing:

$$[[X^1, X^2], X^3] + \text{cyclic} = \pi^2 f^{123} l_s^3 . \quad (97)$$

In fact, this non-associativity is a general property of the algebra displayed in eq.(95), which always implies that

$$[[x^i, x^j], x^k] + \text{cyclic} = 3 c \epsilon^{ijk} . \quad (98)$$

These issues and their interpretations will be further investigated in [44].

So, whereas open string non-commutative geometry is associative and is related to associative open string field theory [5], the closed string geometry could be also related to the non-associativity found in closed string field theory [45, 46], as well as in the more recent doubled closed string field theory of [15, 16, 17, 18].

- Note that T-dual geometry of the twisted torus is actually not a geometrical space in the usual sense, it is rather a T-fold, since the transition functions of global diffeomorphisms (in our example $x^3 \rightarrow x^3 + 2\pi$) are not given by usual $SO(d)$ transformations but by more general duality transformations in $SO(d, d; \mathbb{Z})$ (in our example $\rho \rightarrow 1/\rho$). May be that the emergence of the non-commutative geometry is also related to the fact that we are dealing with non-geometrical compactifications (the T-fold).
- In a given duality frame, we can always express the non-commutative geometry by non-vanishing commutation relations between some space coordinates X^i and some other dual space coordinates \tilde{X}^j ,

$$[X^i, \tilde{X}^j] = i\Theta^{ij} \sim ip^k . \quad (99)$$

In fact, the simultaneous existence of coordinates and dual coordinates, i.e. the introduction of doubled geometry is essential for the emergence of closed string non-commutative geometry in the form we have discussed it in this paper.

- What about closed string non-commutative gravity? Recall, for open strings on D-branes, the non-commutativity is first determined by the closed B -field background. In addition, the open gauge F -field background also contributes to the open string non-commutativity on the D-branes. The non-commutative gauge theory [9, 47, 48, 49] on the D-branes with F -field background has a mathematically very interesting structure, it is determined by the Moyal-Weyl star-product. Via the the Seiberg-Witten (SW) map [9], the non-commutative gauge theory can be mapped to a commutative gauge theory, similar to the T-duality between the commutative torus (D-branes at angles) and the non-commutative torus (D-branes with F -flux). In fact, the (Abelian) gauge theory can be described by the non-linear Born-Infeld action, which contains an infinite number of higher order term in F . Could something analogous be also true for non-commutative closed string geometry, could there be an analogous SW map, which maps a non-commutative (and perhaps non associative) gravity theory to a commutative gravity theory? For that to work, the closed string non-commutativity should not only depend on the H -field, but for curved spaces also on the curvature of the space geometry itself (in analogy to the F -field dependence of the open string non-commutativity). In this case a non-commutative version of gravity along the lines of [50] could be perhaps formulated. Via T-duality resp. via the closed string SW map, non-commutative gravity could be possibly mapped to a gravity theory on a commutative space, which then contains an infinite number of higher order curvature terms (in analogy to the gauge Born-Infeld action).
- In general, six-dimensional, supersymmetric H -flux backgrounds as well as the non-geometric backgrounds can be best described in terms of so-called generalized geometry, which generalizes the $SU(3)$ group structures of the geometric spaces to background with generalized $SU(3) \times SU(3)$ group structures [51, 52, 53, 54, 55, 56, 57, 58, 59, 60]. In analogy to doubled geometry, in case of generalized group structure backgrounds one considers the direct sum of the tangent space and the co-tangent space: $T \oplus T^*$. It follows that the $SU(3, 3)$ duality group has a natural action on spaces with generalized group structure. T-duality is replaced by mirror symmetry, and the symplectic structure is exchanged with the complex structure and vice versa under T-duality. So it would be interesting to see if non-commutative geometry arises for $SU(3) \times SU(3)$ group structure backgrounds in a natural way (see also [61]).

- Finally note that the string picture, where the closed string space geometry cannot be dissolved beyond the string scale due to T-duality and closed string non-commutativity, is complementary to the recent work of Dvali and Gomez [62]. They argue that in pure Einstein gravity the production of black holes sets the shortest possible dissolvable distance, which is just the Planck length.

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Appendix

In this appendix we like to argue that closed string momentum and winding modes define in some sense D-branes in closed string theory. The basic idea is to compare the closed string periodicity conditions with open string boundary conditions at the two ends of the open string.

Open strings:

Let us first consider the open string and briefly recall how non-commutativity arises for open strings in a non-trivial \mathcal{F} -field background. First consider an open string $X(\tau, \sigma)$ in one spatial direction. It has Neumann (N) boundary conditions if at its ends $\sigma = 0, \pi$ the open string coordinate obeys:

$$N : \quad \partial_\sigma X = 0 . \quad (100)$$

On the other hand, Dirichlet (D) boundary conditions mean that at $\sigma = 0, \pi$ the open string has the following boundary conditions:

$$D : \quad \partial_\tau X = 0 . \quad (101)$$

Following the notation of [63], we next consider a 2-dimensional torus with coordinates X_1 and X_2 , where we start with two D1-branes intersecting at an arbitrary angle $\theta_2 - \theta_1$. The open string boundary condition for a strings that

ends on the first D1-brane are at $\sigma = 0$

$$\begin{aligned} N : \quad \partial_\sigma X_1 + \tan \theta_1 \partial_\sigma X_2 &= 0, \\ D : \quad \partial_\tau X_2 - \tan \theta_1 \partial_\tau X_1 &= 0. \end{aligned} \quad (102)$$

Hence for $\theta_1 = 0$ the open string has Neumann boundary conditions in the X_1 direction and Dirichlet boundary conditions in X_2 , i.e. the D1-brane is along the X_1 axis. For $\theta_1 = \pi/2$ the situation is reversed.

At $\sigma = \pi$ we either require the same boundary conditions in case the string also ends the first D1-brane. However if at $\sigma = \pi$ the open string ends at the second D1-brane, we have to require

$$\begin{aligned} N : \quad \partial_\sigma X_1 + \tan \theta_2 \partial_\sigma X_2, &= 0, \\ D : \quad \partial_\tau X_2 - \tan \theta_2 \partial_\tau X_1 &= 0, \end{aligned} \quad (103)$$

Introducing complex coordinates, $X = X^1 + iX^2$, the D1 boundary conditions for the branes at angles can be also written as (at $\sigma = 0, \pi$)

$$\begin{aligned} N : \quad e^{-\theta_i} \partial_\sigma X + e^{i\theta_i} \partial_\sigma \bar{X}, &= 0, \\ D : \quad e^{-\theta_i} \partial_\tau X - e^{i\theta_i} \partial_\tau \bar{X}, &= 0. \end{aligned} \quad (104)$$

The mode expansion satisfying these two boundary conditions (102) and (103) for an open string with its ends on the two different branes contains twisted oscillators $\alpha_{n+\nu}$ and looks like

$$\begin{aligned} X_1 = x_1 &+ i\sqrt{\alpha'} \sum_{n \in \mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \cos[(n+\nu)\sigma + \theta_1] + \\ &i\sqrt{\alpha'} \sum_{m \in \mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \cos[(m-\nu)\sigma - \theta_1], \\ X_2 = x_2 &+ i\sqrt{\alpha'} \sum_{n \in \mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \theta_1] - \\ &i\sqrt{\alpha'} \sum_{m \in \mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \theta_1], \end{aligned} \quad (105)$$

with $\nu = (\theta_2 - \theta_1)/\pi$. (Note that for $\nu \neq 0$ this mode expansion is analogous to the mode expansion of a string in a twisted sector of an orbifold.) Using the usual commutation relation

$$[\alpha_{n+\nu}, \alpha_{m-\nu}] = (n+\nu) \delta_{m+n,0} \quad (106)$$

and the vanishing of the commutator of the center of mass coordinates x_1 and x_2 one can easily show that for D-branes at angles the general equal time commutator vanishes

$$[X_i(\tau, \sigma), X_j(\tau, \sigma')] = 0. \quad (107)$$

Therefore, the geometry of D-branes at angles, but without background gauge fields, is always commutative.

Performing open string T-duality in the X_1 direction implies exchanging the world sheet coordinates τ and σ for X_1 . The two intersecting D1-branes will become two D2-branes, and the geometrical intersection angles θ_1 and θ_2 of the two D1-branes now correspond to constant gauge field strength background $\mathcal{F}_{1,2}$ -fields for D2-branes on the dual torus ($\mathcal{F}_{1,2} = \cot \theta_{1,2}$). Hence one gets the two mixed boundary conditions for the open strings

$$\begin{aligned}\partial_\sigma X_1 + \cot \theta_1 \partial_\tau X_2 &= 0, \\ \partial_\sigma X_2 - \cot \theta_1 \partial_\tau X_1 &= 0\end{aligned}\tag{108}$$

at $\sigma = 0$ and

$$\begin{aligned}\partial_\sigma X_1 + \cot \theta_2 \partial_\tau X_2 &= 0, \\ \partial_\sigma X_2 - \cot \theta_2 \partial_\tau X_1 &= 0\end{aligned}\tag{109}$$

at $\sigma = \pi$. Again it is easy to write these mixed boundary conditions in terms of the complex coordinate $X = X^1 + iX^2$. The mode expansion satisfying these boundary conditions is

$$\begin{aligned}X_1 = x_1 &- \sqrt{\alpha'} \sum_{n \in \mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \theta_1] - \\ &\sqrt{\alpha'} \sum_{m \in \mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \theta_1], \\ X_2 = x_2 &+ i\sqrt{\alpha'} \sum_{n \in \mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \theta_1] - \\ &i\sqrt{\alpha'} \sum_{m \in \mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \theta_1].\end{aligned}\tag{110}$$

Now one can compute the commutator

$$[X_1(\tau, \sigma), X_2(\tau, \sigma')] = [x_1, x_2] + i \sum_{n \in \mathbb{Z}} \frac{2\alpha'}{n+\nu} \sin[(n+\nu)\sigma + \theta_1] \sin[(n+\nu)\sigma' + \theta_1].\tag{111}$$

As has been shown in [64, 65, 63], for $\sigma = \sigma' = 0$ the evaluation of the sum in (111) yields

$$[X_1(\tau, 0), X_2(\tau, 0)] = -\frac{2\pi i \alpha' \mathcal{F}_1}{1 + \mathcal{F}_1^2}\tag{112}$$

and for $\sigma = \sigma' = \pi$ one analogously obtains

$$[X_1(\tau, \pi), X_2(\tau, \pi)] = \frac{2\pi i \alpha' \mathcal{F}_2}{1 + \mathcal{F}_2^2}.\tag{113}$$

Thus, the coordinates only non-commute at the boundary of the world-sheet, where the commutator can be expressed entirely in terms of the gauge field on the local D-brane. Only for $\theta_1 = \theta_2 \in \{0, \pi/2\}$ the X_1 and X_2 coordinates commute on the D-brane, in all other cases the end points see a non-commuting space-time. Moreover, in the compact case for rational D-branes the non-commutative theory on the D-branes is mapped via T-duality to a commutative theory on D-branes at angles. This is only a special example of the more general rule pointed out in [9] that for rational points the non-commutative torus is T-dual to a commutative one.

Closed strings:

First consider a closed string on a circle, which is a pure momentum state, i.e. has vanishing winding number $N = 0$. The momentum states propagate in X -space; they obey the standard periodicity conditions along the closed string in σ -direction. We regard these momentum string boundary conditions to be analogous to the open string Neumann boundary conditions eq.(100) and hence call them closed string "Neumann" conditions in X -space:

$$\begin{aligned} "N" : \quad X(\tau + 2\pi, \sigma) &= X(\tau, \sigma) + 2\pi M/R \\ X(\tau, \sigma + 2\pi) &= X(\tau, \sigma). \end{aligned} \quad (114)$$

On the other hand, a momentum state does not propagate in the dual space; hence it possesses "Dirichlet" boundary conditions in the \tilde{X} -direction:

$$\begin{aligned} "D" : \quad \tilde{X}(\tau + 2\pi, \sigma) &= \tilde{X}(\tau, \sigma), \\ \tilde{X}(\tau, \sigma + 2\pi) &= \tilde{X}(\tau, \sigma) + 2\pi M/R \end{aligned} \quad (115)$$

Second, a pure winding state with vanishing momentum number $M = 0$ does not propagate in X -space, and it satisfies the following closed string "Dirichlet" boundary conditions in X -space and "Neumann" boundary conditions in \tilde{X} -space:

$$\begin{aligned} "N" : \quad \tilde{X}(\tau + 2\pi, \sigma) &= \tilde{X}(\tau, \sigma) + 2\pi NR, \\ \tilde{X}(\tau, \sigma + 2\pi) &= \tilde{X}(\tau, \sigma), \\ "D" : \quad X(\tau + 2\pi, \sigma) &= X(\tau, \sigma), \\ X(\tau, \sigma + 2\pi) &= X(\tau, \sigma) + 2\pi NR. \end{aligned} \quad (116)$$

In view of these closed string boundary conditions, one can loosely say that the momentum states propagate on the a kind of "D1-brane" in the doubled (X, \tilde{X}) -space, where the "D1-brane" is along the X -direction. On the other hand, a pure winding state satisfied "Neumann" boundary conditions along \tilde{X} . Therefore winding states live on a kind of "D1-brane" along \tilde{X} . T-duality exchanges a "D1-brane" in X -direction with a "D1-brane" in \tilde{X} -direction.⁶

⁶Of course, it would be amusing to speculate that in doubled gravity these "D-branes" arise as dynamical solutions.

We can also extend the discussion about the closed string "D-branes" and their boundary conditions for the toroidal compactifications. To be specific, we consider the closed string boundary of the twisted torus compactification discussed before. The boundary conditions with respect to $\sigma \rightarrow \sigma + 2\pi$ were already worked out in section 3 (see eq.(63)). Now we include also the periodicity conditions with respect to $\tau \rightarrow \tau + 2\pi$. Then we obtain for the complex closed string coordinates $X(\tau, \sigma) = X_L(\tau + \sigma) + X_R(\tau - \sigma)$ as well as for its dual coordinate (T-duality in x^1 -direction) $\tilde{X}(\tau, \sigma) = \tilde{X}_L(\tau + \sigma) + \tilde{X}_R(\tau - \sigma)$ the following set of boundary conditions:

$$\begin{aligned} "N" : \quad X(\tau, \sigma + 2\pi) &= e^{i\theta} X(\tau, \sigma), \\ "D" : \quad \tilde{X}(\tau + 2\pi, \sigma) &= e^{i\theta} \tilde{X}(\tau, \sigma). \end{aligned} \quad (117)$$

So we see that the momentum states satisfy rotated (i.e. twisted) "Neumann" boundary conditions in the 2-dimensional X -space and "Dirichlet" boundary conditions in the 2-dimensional \tilde{X} -space. Of course, for the dual H -field background the "Neumann" and "Dirichlet" directions are exchanged:

$$\begin{aligned} "N" : \quad \tilde{X}(\tau, \sigma + 2\pi) &= e^{i\theta} \tilde{X}(\tau, \sigma), \\ "D" : \quad X(\tau + 2\pi, \sigma) &= e^{i\theta} X(\tau, \sigma). \end{aligned} \quad (118)$$

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